Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-10-13

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

$$\int_{\theta_0}^{\theta_1} \int_0^{f(\theta)} r \, \mathrm{d}r \, \mathrm{d}\theta = \int_{\theta_0}^{\theta_1} \frac{1}{2} f(\theta)^2 \, \mathrm{d}\theta.$$

On the left we have the §15.3 method (note that computing an area of a region is just integrating the function 1 over that region), and on the right we have the old formula.

Answers to computations

Problem 1. Since the second one is obtained from the first by replacing all instances of x with $x+1$, that means the second region is just the first region translated by 1 unit in the negative x direction. In particular, they will have the same area.

(a) For the first region, we integrate

$$\iint_{x^2+y^2 \le 2x} \left((2x) - (x^2+y^2)\right) \mathrm{d}x \, \mathrm{d}y.$$

The region $x^2 + y^2 \le 2x$ can be parametrized using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ as

$$0 \le r \le 2\cos\theta, -\pi/2 \le \theta \le \pi/2$$

hence in polar we get the integral

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (2r\cos\theta - r^2) r \, \mathrm{d}r \, \mathrm{d}\theta.$$

(b) For the second region, proceeding in exactly the same way, we get the integral

$$\iint_{(x+1)^2+y^2 \le 2(x+1)} \left(2(x+1) - ((x+1)^2 + y^2)\right) \mathrm{d}x \, \mathrm{d}y$$

which simplifies to

$$\iint_{x^2+y^2 \le 1} (1 - x^2 - y^2) \, \mathrm{d}x \, \mathrm{d}y$$

and in polar this is just

$$\int_0^{2\pi} \int_0^1 (1 - r^2) r \, \mathrm{d}r \, \mathrm{d}\theta$$

Both integrals work out to $\boxed{\pi/2}$ with the latter one being simpler to compute.

Problem 2.

(a) The expression is equal to

$$\lim_{a\to\infty} \int_0^{2\pi} \int_0^a e^{-r^2} r \, \mathrm{d}r \, \mathrm{d}\theta = \lim_{a\to\infty} \pi\left(1 - e^{-a^2}\right) = \boxed{\pi}.$$

(b) Observe that D_b is contained inside of S_b, which is in turn contained inside of $D_{\sqrt{2}b}$. Since $e^{-x^2-y^2} > 0$ always, we conclude that

$$\iint_{D_b} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y \le \iint_{S_b} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y \le \iint_{D_{\sqrt{2}b}} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y.$$

As $b \to \infty$, the leftmost expression tends to π by (a). So does the rightmost expression, because $\sqrt{2}b \to \infty$ as $b \to \infty$. Therefore, by the Squeeze Theorem, when $b \to \infty$ the middle expression converges to π as well.

(c) This is related to (b) as follows.

$$\begin{aligned}\lim_{a\to\infty}\iint_{S_a} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y &= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,\mathrm{d}x \int_{-a}^{a} e^{-y^2}\,\mathrm{d}y\right)\\ &= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,\mathrm{d}x\right)^2\\ &= \left(\lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,\mathrm{d}x\right)^2\\ &= I^2\end{aligned}$$

thus $I = \boxed{\sqrt{\pi}}$.

(d) Clearly it is always nonnegative (it is strictly positive). Verifying that it integrates to 1 is just single-variable calculus. Apply the change of variables $x = \sqrt{2}u$:

$$\lim_{a\to\infty}\int_{-a/\sqrt{2}}^{a/\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-u^2}\sqrt{2}\,\mathrm{d}u = \frac{1}{\sqrt{\pi}}\lim_{b\to\infty}\int_{-b}^{b} e^{-u^2}\,\mathrm{d}u = \frac{I}{\sqrt{\pi}} = 1$$

where $b = a/\sqrt{2}$. Note that $b \to \infty$ as $a \to \infty$.